AIAA 80-0239R

Sequential Orbit Determination with Auto-Correlated Gravity Modeling Errors

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A sequential orbit determination algorithm is given which models auto-correlated gravity errors for near-circular orbits. Using results from geodesy theory it is demonstrated that gravity modeling error auto-correlation cannot be approximated with the Kalman white process noise model. The resulting auto-correlated orbit error process is transformed from the status of non-Markov to Markov at the expense of an assumption which imposes symmetries on certain covariance functions. It is emphasized that the character of the trajectory estimation problem is significantly changed by recognizing and using the auto-correlation property associated with gravity modeling errors.

I. Introduction

SEQUENTIAL orbit determination procedures enjoy significant operational advantages over the older methods[1] of batch least-squares differential corrections: ephemeris integration across overlapping measurement time intervals is eliminated thereby significantly reducing computation time, force modeling discontinuities are easily bridged with a significant reduction in the time required to remove transients due to uncertainties associated with the discontinuities, and the extended Kalman filter provides a structure to formally account for stochastic force modeling errors.

With the richer structure associated with sequential procedures, one would have hoped to say that they enjoy a consistent and significant accuracy improvement over the older batch methods. In the author's experience this has not been realized.

Herein the author will attempt to show that a fundamental reason for this disappointment is to be found in the failure of stochastic force modeling errors to satisfy the white process noise hypothesis of the Kalman filter. Particularly we will show in Sec. V that gravity modeling errors are auto-correlated with respect to time so that any white noise approximation will yield a nonoptimal procedure.

The problem of interest is stated in Sec. III, using the notation and definitions introduced in Sec. II. An approximate solution which accounts for gravity modeling error auto-correlation is given in Sec. IV. The remaining sections (V-IX) are devoted to the derivation of the solution given in Sec. IV. Equation (6), given in the solution, follows Eq. (26) in Sec. IX from a derivational point of view.

II. State Space and Measurement Space

The geocentric orbit estimation problem can be initially characterized as a dynamical system with inputs, outputs, and that which relates inputs to outputs, the state. Most significantly, inputs are forces: Earth gravity, lunar and solar gravity, solar radiation pressure, and forces due to air density interacting with the moving surfaces of the space vehicle. The output consists of measurements, primarily tracking data. Then the state consists of those parameters or functions required to explain or predict the output measurements in terms of the input forces. In addition to osculating orbit element functions (or instantaneous position and velocity functions), the state-space will include some subset of force modeling functions and some subset of measurement modeling functions required to complete the explanation of output measurements in terms of input forces. Since the state is not directly measured, it must be estimated in order to represent the orbit in the past, present or future.

Our interest is focused on the role of gravity modeling errors, how they relate and contribute to the orbit error covariance function, and how they affect the orbit estimation process. Therefore, we shall restrict our attention to a six-dimensional substate, the orbit; forces other than Earth gravity will be ignored.

Out substate will be denoted by a 6×1 column matrix $z(t)$ of osculating orbital element time functions. Its propagation with time, referred to as ephemeris integration, is necessarily nonlinear:

$$z(t_{k+1})=\phi\left(t_{k+1};z(t_k),t_k,u(\tau)\right)$$

where $t_k\le\tau\le t_{k+1}$, and $u(\tau)$ is a gravity modeling function.

The measurement value at time t_j, an $m\times 1$ column matrix, will be denoted y_j. The measurement representation for time t_j is a nonlinear function of the orbit at time t_j and will be denoted $y(z(t_j))$, an $m\times 1$ column matrix of functions.

An orbit estimate will be denoted

$$\hat{z}_{k|j}\equiv\hat{z}(t_k|t_j)$$

meaning that it has epoch (time tag) at t_k and reflects all measurement data through time t_j. Similarly, the true orbit is denoted

$$z_k\equiv z(t_k)$$

The term "measurement residual" is used to denote the difference

$$y_{j+1}-y(\hat{z}_{j+1|j})$$

A. Orbit Errors

The orbit error defined by

$$\Delta z_{k|j}=z_k-\hat{z}_{k|j} \tag{1}$$

is multivariate Gaussian distributed with zero mean. Its covariance function is given by

$$P_{k|j}=\mathrm{E}\{\Delta z_{k|j}\Delta z_{k|j}{}^T\} \tag{2}$$

Presented as Paper 80-0239 at the AIAA 18th Aerospace Sciences Meeting, Pasadena, Calif., Jan. 14-16, 1980; submitted Jan. 17, 1980; revision received Aug. 27, 1980.

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Introduction of an orbit error model is deferred until Sec. VII.

B. Measurement Residual Errors

The measurement residual is assumed to have the form

$$y_{j+1}-y(\hat{z}_{j+1|j})=\frac{\partial y(z)}{\partial z}\bigg|_{\hat{z}_{j+1}}\Delta z_{j+1|j}+\nu_{j+1} \qquad (3a)$$

where ν_j is multivariate Gaussian distributed with zero mean, uncorrelated with respect to time, and uncorrelated with gravity modeling errors.

The measurement error covariance function is given by

$$\mathrm{E}\{\nu_j\nu_k^T\}=R_j\delta_{jk} \qquad (3b)$$

where δ_{jk} is the Dirac delta function.

III. The Problem

Given values for the optimal orbit estimate $\hat{z}_{k|k}$ and orbit error covariance $P_{k|k}$ at time t_k, and given a new measurement value y_{k+1} and measurement error covariance R_{k+1} at time t_{k+1} determine the optimal orbit estimate $\hat{z}_{k+1|k+1}$ and orbit error covariance $P_{k+1|k+1}$ for time t_{k+1}; where force modeling errors are due only to gravity modeling errors, and orbit eccentricity is small.

IV. An Approximate Solution

In Sec. V it will be shown that the 3×3 covariance function on gravity acceleration errors $\Delta u(t)$ can be structured

$$C(t,t-\tau)=\mathrm{E}\{\Delta u(t)\Delta u(\tau)^T\}=C(t,0)\rho(t-\tau) \qquad (4)$$

where $\Delta u(t)$ is assumed to be Gaussian distributed, $C(t,t-\tau)$ is diagonal, and $\rho(t-\tau)$ is a diagonal auto-correlation function.

First, calculate the constant 3×3 diagonal auto-correlation integral at perigee (see Fig. 1 for an example):

$$\int_{-\bar{\gamma}}^{\bar{\gamma}}\rho(\gamma)\mathrm{d}\gamma \qquad \frac{2}{9}\le\bar{\gamma}/(\text{orbit period})\le\frac{4}{9}$$

and store it. The following sequence is then calculated recursively.

1) Use ephemeris integration to obtain the optimal predicted orbit estimate for time t_{k+1}:

$$\hat{z}_{k+1|k}=\phi(t_{k+1};\hat{z}_{k|k},t_k,u_\gamma) \qquad t_k\le\gamma\le t_{k+1} \qquad (5)$$

2) Use Riemann partition ($\Delta\tau\le$period/36) and integration to construct the approximate gravity modeling error covariance matrix:

$$P_{k+1,k}^{\iint}\simeq\int_{t_k}^{t_{k+1}}\Phi(t_{k+1},\tau)G(\tau)C(\tau,0)\times\left[\int_{-\bar{\gamma}}^{\bar{\gamma}}\rho(\gamma)\mathrm{d}\gamma\right]G(\tau)^T\Phi(t_{k+1},\tau)^T\mathrm{d}\tau \qquad (6)$$

where $\Phi(t,\tau)$ is a 6×6 linear orbit error transition matrix, $G(\tau)$ is the 6×3 jacobian matrix of partial derivatives found in the Lagrange-Gaussian equations of motion for orbital elements.[2,3] *Note:* Eq. (6) is derived from Eq. (26) given in Sec. IX.

3) Calculate the predicted estimate of the orbit error covariance matrix for time t_{k+1}:

$$P_{k+1|k}=\Phi_{k+1,k}P_{k|k}\Phi_{k+1,k}^T+P_{k+1,k}^{\iint} \qquad (7)$$

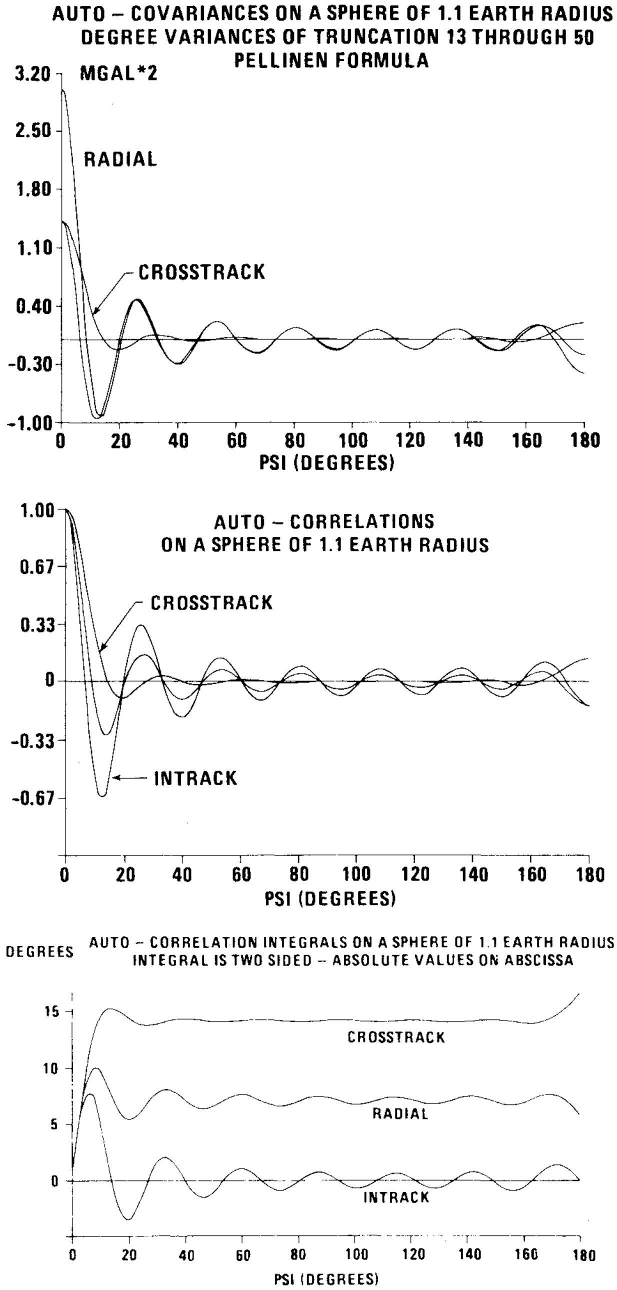

Fig. 1 Gravity error auto-correlations.

4) Represent the measurement for time t_{k+1}:

$$y(\hat{z}_{k+1|k})$$

5) Calculate the measurement—orbit jacobian matrix of partial derivatives:

$$H_{k+1}=\frac{\partial y(z)}{\partial z}\bigg|_{\hat{z}_{k+1|k}} \qquad (8)$$

6) Estimate the orbit correction:

$$\Delta\hat{z}_{k+1|k+1}=P_{k+1|k}H_{k+1}^T[H_{k+1}P_{k+1|k}H_{k+1}^T+R_{k+1}]^{-1}\times(y_{k+1}-y(\hat{z}_{k+1|k}))$$

due to the measurement y_{k+1}.

7) Correct the orbit estimate:

$$\hat{z}_{k+1|k+1}=\hat{z}_{k+1|k}+\Delta\hat{z}_{k+1|k+1} \qquad (9)$$

8) Update the orbit error covariance matrix:

$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k}H_{k+1}^T[H_{k+1}P_{k+1|k}H_{k+1}^T + R_{k+1}]^{-1}H_{k+1}P_{k+1|k} \quad (10)$$

Now given the next measurement y_{k+2}, the above sequence of calculations is repeated after replacing k with $k+1$.

The reader will notice that the procedure given would be equivalent to an extended Kalman filter were it not for the introduction of the new object $P_{k+1,k}^{ff}$ into the computation for $P_{k+1|k}$ in place of Kalman's Q_{k+1}.[4]

This difference is significant, for gravity modeling errors can be accommodated with the structure $C(t,0)\rho(t-\tau)$, whereas the white process noise hypothesis requires the integral on $\rho(t-\tau)$ to be a 3×3 identity matrix, which seriously violates the facts, as shown in the next section.

V. Gravity Error Covariance on a Sphere

Consider first a circular reference orbit of radius r containing a space vehicle at orbital latitude $u=u(t)$. Let Ψ be a displacement in central angle measured in the orbit plane such that the orbital latitude at time τ is given by

$$u(\tau) = u(t) + \Psi \quad (11)$$

Then Ψ is a function of t and τ; and since $u(\tau+T) - u(t+T) = u(\tau) - u(t)$ for any T, then Ψ can be written

$$\Psi \equiv \Psi(t-\tau)$$

Let $\Delta g(t)$ be a random gravity error vector with radial, intrack, and cross-track components $\Delta g_R(t)$, $\Delta g_I(t)$, and $\Delta g_C(t)$ when resolved on the Gaussian orthonormal trajectory vector basis. Following Kaula,[5]† define a degree variance $\sigma_T^2(n)$ of harmonic expansion truncation (i.e., omission) such that

$$\sigma_T^2(n) = \sum_{m=0}^{n} (C_{nm}^2 + S_{nm}^2) \quad (12)$$

where C_{nm} and S_{nm} are geopotential coefficients which will be ignored in the ephemeris integration. And define a different kind of degree variance

$$\sigma_e^2(n) = \sum_{m=0}^{n} [\sigma_c(\Delta C_{nm})^2 + \sigma_s(\Delta S_{nm})^2] \quad (13)$$

to account for coefficient errors of commission of those coefficients which will be used in the ephemeris integration.

Finally, define

$$\sigma(n)^2 = \sigma_T^2(n) \quad \text{if } C_{nm} \text{ and } S_{nm}\ (m=0,1,\ldots,n) \text{ are not used in the ephemeris integration}$$

$$= \sigma_e^2(n) \quad \text{if } C_{nm} \text{ and } S_{nm} \text{ are used in the ephemeris integration} \quad (14)$$

Approximations by Kaula,[5,6] and Gersten et al.[7] have been given for the gravity error covariances

$$C_\alpha(t,t-\tau) = \mathrm{E}\{\Delta g_\alpha(t)\Delta g_\alpha(\tau)\} \qquad \alpha=R,I,C \quad (15)$$

in terms of degree variances, where the expectation E { · } has been taken over a sphere. They are

$$C_R(t,t-\tau) = \sum_{n=2}^{\infty}\left(\frac{a_e}{r(t)}\right)^{2n+4}\sigma(n)^2 P_n[\cos\Psi(t-\tau)] \quad (16)$$

†The geodesists' "anomalous gravity"[6] is analogous to our "ephemeris gravity."

$$C_I(t,t-\tau) = \sum_{n=2}^{\infty}\frac{n(n+1)}{2(n-1)^2}\left(\frac{a_e}{r(t)}\right)^{2n+4}\sigma(n)^2 \times\left[P_n[\cos\Psi(t-\tau)] - \frac{P_{n2}[\cos\Psi(t-\tau)]}{n(n+1)}\right] \quad (17)$$

$$C_c(t,t-\tau) = \sum_{n=2}^{\infty}\frac{n(n+1)}{2(n-1)^2}\left(\frac{a_e}{r(t)}\right)^{2n+4}\sigma(n)^2 \times\left[P_n[\cos\Psi(t-\tau)] + \frac{P_{n-1,2}[\cos\Psi(t-\tau)]}{n(n+1)}\right] \quad (18)$$

where a_e is the radius of the Earth's equator, $P_n(\cos\Psi)$ is a Legendre polynomial of degree n, and $P_{n2}(\cos\Psi)$ is an associated Legendre function of degree n and order 2. Here the first argument t of $C_\alpha(t,t-\tau)$ is superfluous due to the constant property of r on a sphere; it will be used later for noncircular orbit approximations.

Construct the diagonal auto-covariance function on gravity modeling errors:

$$C(t,t-\tau) = \begin{bmatrix} C_R(t,t-\tau) & 0 & 0 \\ 0 & C_I(t,t-\tau) & 0 \\ 0 & 0 & C_C(t,t-\tau) \end{bmatrix} \quad (19)$$

Now $t-\tau=0$ implies $\Psi=0$ so that the Legendre polynomials are unity and the associated Legendre functions are zero when $\tau=t$. The instantaneous variance matrix $C(t,0)$ has nonzero diagonal elements which enables the definition of a diagonal auto-correlation function:

$$\rho(t-\tau) = C(t,0)^{-1}C(t,t-\tau) \quad (20)$$

Integrals on the diagonal elements $\rho_\alpha(t-\tau)$, $\alpha=R,I,C$, have important properties: $\int\rho_R(\gamma)d\gamma$ and $\int\rho_C(\gamma)d\gamma$ generate significant magnifications of the instantaneous variances $C_R(t,0)$ and $C_C(t,0)$ when $C(t,\gamma)$ is integrated with respect to γ, whereas $\int\rho_I(\gamma)d\gamma$ is essentially zero. These properties were determined by numerical experiments; an example is given in Fig. 1. Inspection of Eqs. (17) and (18) reveals that $C_I(t,0)=C_C(t,0)$. Thus the effect of auto-correlation drastically modifies the input instantaneous gravity error variances.

The auto-correlation function associated with a white noise process (unit variance) is simply a positive unit spike at the origin with zero elsewhere. This is very different from any of the three functions defined by Eq. (20). The center graph of Fig. 1 presents an example discussed below.

It is thus seen that the auto-correlation functions $\rho_\alpha(t-\tau)$ and their integrals bear no resemblance to Dirac delta functions and their integrals. Therefore the white process noise Kalman filter hypothesis must be rejected for use in sequential orbit determination.

Consider a particular case. The ephemeris is integrated using a geopotential expansion which is complete only through degree and order twelve; and such that linearization is appropriate on a sphere of radius 1.1 Earth radius. Pellinen's[8] gravity error degree variances of truncation generate the graphs of Fig. 1.

VI. Noncircular Orbits with Small Eccentricity

The factorization $C(t,t-\tau) = C(t,0)\rho(t-\tau)$ yields a property which enables the application of Eqs. (16-18) to noncircular orbits with small eccentricity: $C(t,0)$ is sensitive to small variations in r whereas $\rho(t-\tau)$ is not. This property was determined by comparing numerical experiments. Then represent the trajectory as a passage through many spheres with different radii, but evaluate $\rho(t-\tau)$ only once on the sphere at perigee. Treating $\rho(t-\tau)$ as otherwise independent of $r(t)$, the use of $C(t,0)\rho(t-\tau)$ provides an excellent approximation to the truth: dynamics are captured by evaluating $C(t,0)$ as a function of the variable $r(t)$ around the orbit, the approximation is exact at perigee where $C(t,0)$ is maximum, and it is otherwise insensitive to errors in the auto-correlation approximation $\rho(t-\tau)$.

The first argument t of $C(t,t-\tau)$ is now required by the variations in $r=r(t)$.

VII. The Orbit Error Model

After appropriate linearizations the orbit error Δz can be shown to obey, approximately, the ordinary linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta z(t)=F(t)\Delta z(t)+G(t)\Delta u(t) \tag{21}$$

where

$F(t)\equiv\tilde{F}(t,\tilde{z}(t))$ 6×6 system matrix

$G(t)\equiv\tilde{G}(t,\tilde{z}(t))$ 6×3 matrix of partial derivatives

$\Delta u(t)\equiv\Delta\tilde{u}(t,\tilde{z}(t))$ 3×1 matrix of random gravity errors

$\tilde{z}(t)$ is a reference orbit, periodically rectified

$\Delta z(t_0)=\Delta z_0$ initial orbit error

Given a choice of osculating orbital elements for $z(t)$, the detailed structures for $F(t)$ and $G(t)$ can be determined with the aid of the theory of variation of parameters—from celestial mechanics.[3,9]

The random gravity error $\Delta u(t)=(\Delta g_R(t),\ \Delta g_I(t),\ \Delta g_c(t))^T$ is assumed to be Gaussian distributed. Using the orthogonal properties of Legendre functions it is easily shown to be unbiased.[6] In the previous section it was shown to be strongly auto-correlated in time.

Equation (21) has the well-known integral[10]

$$\Delta z(t_{k+1})=\Phi(t_{k+1},t_k)\Delta z(t_k)+\int_{t_k}^{t_{k+1}}\Phi(t_{k+1},\tau)G(\tau)\Delta u(\tau)\mathrm{d}\tau \tag{22}$$

where $t_k<t_{k+1}$. Evolution of the orbit error $\Delta z(t)$ according to Eq. (22) is formally non-Markov, as will now be shown.

Let the current time be $t_j=t_{k+1}$ and a future time of interest be $t_{j+1}=t_{k+2}\,(t_j<t_{j+1})$. Given the orbit error at current time $\Delta z(t_j)$ and its covariance $\mathrm{E}\{\Delta z(t_j)\Delta z(t_j)^T\}$ consider the orbit error at a future time $\Delta z(t_{j+1})$. Now if no additional data concerning orbit errors $\Delta z(\tau)$ at previous times $\tau<t_j$ can alter the conditional probability for the orbit error at any future time then the orbit error process is said to be Markov[11]; otherwise it is non-Markov.

The linear mapping of the Gaussian process $\Delta u(\tau)$ into $\Delta z(t)$ given by Eq. (22) generates a Gaussian orbit error process completely captured by its mean and covariance. The mean is zero. Consider the covariance

$$P(t_{j+1})\equiv\mathrm{E}\{\Delta z(t_{j+1})\Delta z(t_{j+1})^T\} \tag{23}$$

$$=P_j(t_{j+1})+P_{j+1}(t_{j+1})+P_L(t_{j+1})+P_R(t_{j+1}) \tag{24}$$

where each term of Eq. (24) is defined by inserting Eq. (22) into Eq. (23) and associating with the integration domain defined by Fig. 2.

Now the covariance $P_L(t_{j+1})$ is an integral on the product of $\Delta z(t_j)$ and $\Delta u(\eta)$ where $t_j\leq\eta\leq t_{j+1}$. But $\Delta z(t_j)$ depends on gravity errors $\Delta u(\tau)$ where $t_{j-1}\leq\tau\leq t_j$ according to Eq. (22). Thus the covariance $P_L(t_{j+1})$ for the future time t_{j+1} depends on gravity errors $\Delta u(\tau)$ in the past $(\tau<t_j)$. Therefore the conditional probability for the orbit error in the future depends on data concerning orbit errors in the past. Evolution of the orbit error is accordingly non-Markov. We wish to remove this difficulty.

Practically, it is important to note that if $P_L(t_{j+1})$ and $P_R(t_{j+1})$ were given, along with $\Delta z(t_j)$ and $P_j(t_j)$, the orbit error process could be called Markov because

$$P_j(t_{j+1})=\Phi(t_{j+1},t_j)P_j(t_j)\Phi(t_{j+1},t_j)^T$$

Alternatively, if $P_L(t_{j+1})$ could be approximated using only data $\Delta u(\tau)$ in the future $(t_j<\tau)$, then the orbit error process could be called Markov—approximately. To this end introduce two assumptions

$$P_L(t_{k+1})\cong P_L(t_k)\qquad P_R(t_{k+1})\cong P_L(t_{k+1})^T$$

$$(t_k<t_{k+1};\ k=0,1,2,\ldots)$$

With these adopted requirements for symmetrical integrals, we shall now refer to the evolution of the orbit error as Markov.

VIII. A New Generalization to the Kalman Filter Theorem

The Gauss-Markov process defined in the previous section is significantly different from that considered by Kalman,[12] but it is a happy fact that the Kalman filter theorem derivation goes through almost the same way as before.[13] Sherman's theorem[13,14] instructs us to associate the optimal estimate of the orbit correction, given measurement residual, with the conditional mean. One then constructs the conditional multivariate Gaussian density function and simply identifies the conditional mean and covariance.[15] An algorithm ensues.[15] An extension to the nonlinear problem (i.e., linearization of the nonlinear problem) defined in Sec. III leads to the solution given in Sec. IV, according to further developments given herein. The only significant difference, from the Kalman filter, is found in the representation of $P(t_{k+1}|t_k)$ defined by

$$P(t_{k+1}|t_k)=\mathrm{E}\{\Delta z(t_{k+1}|t_k)\Delta z(t_{k+1}|t_k)^T\}$$

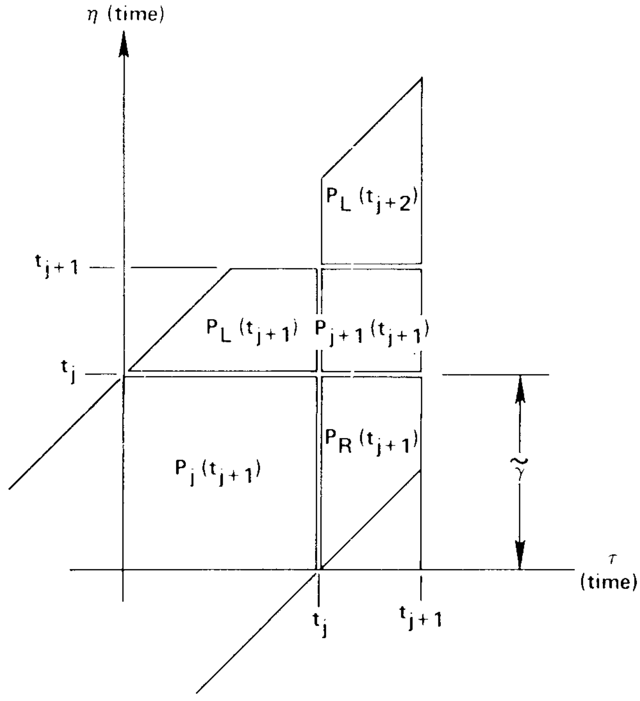

Fig. 2 Integration domain.

where $\Delta z(t_{k+1}|t_k)$ is the orbit error at time t_{k+1} given the last measurement residual at time t_k. This is the subject of the next section.

IX. Orbit Error Covariance Evaluation

Referring to Eq. (24), define

$$P^{JJ}_{k+1,k} = P_{k+1}(t_{k+1}) + P_L(t_{k+1}) + P_R(t_{k+1}) \tag{25}$$

With the aid of the assumptions of integral symmetries given in Sec. VII it is straightforward to show that

$$P^{JJ}_{k+1,K} = \int_{t_k}^{t_{k+1}} H(\tau)C(\tau,0)\left[\int_{-P/2}^{P/2} \rho(\gamma)H(\gamma+\tau)^T \mathrm{d}\gamma\right]\mathrm{d}\tau \tag{26}$$

where $H(t) = \Phi(t_{k+1},t)G(t)$ and P is the orbital period. The bridge which takes us from Eq. (26) to the approximation given by Eq. (6) will now be described.

For a (near) circular orbit the elements of $H(t)$ are either slowly varying functions of time, zero, or they are simple linear combinations of sinusoids which depend on some measure of orbital central angle—say mean argument of orbital latitude $U(t)$ (use mean orbital longitude if $\sin i = 0$). For example, let mean latitude and mean motion $n(t)$ be elements of the orbit $z(t)$. Then the radial contribution to mean orbital latitude and the in-track contribution to mean motion generate two such slowly varying time functions of $H(t)$. At most we are interested in the first two significant decimals of $P^{JJ}_{k+1,k}$ element values because the input error variances are themselves uncertain beyond the first or (perhaps) second decimal. Therefore assume that the slowly moving time functions of $H(t)$ are constant over the integration interval $(-P/2,\ P/2)$—only for the purpose of evaluating $P^{JJ}_{k+1,K}$. Equation (6) follows immediately from Eq. (26) for this subset of elements.

The sinusoids of $H(t)$ are shown to be approximately integrated with Eq. (6) by utilizing two properties. First, all expressions of the form

$$\int_{-P/2}^{P/2} \rho(t)\sin(\Psi(t))\,\mathrm{d}t \tag{27}$$

are zero because $\rho(t)$ is even, according to Eqs. (16-18). This leaves terms with factors of form

$$\int_{-P/2}^{P/2} \rho(t)\cos\Psi(t))\,\mathrm{d}t \tag{28}$$

which are nonzero; and these can be approximated

$$\int_{-P/2}^{P/2} \rho(t)\cos(\Psi(t))\,\mathrm{d}t \cong \int_{-\tilde{\gamma}}^{\tilde{\gamma}} \rho(t)\,\mathrm{d}t \tag{29}$$

using a second property to be described below.

Geopotential coefficients of low degree and order are very well determined.[16] Thus the gravity modeling errors of interest are associated with higher degree harmonics. The integral with respect to central angle Ψ of a higher degree Legendre polynomial or associated Legendre function (degree n) taken symmetrically about zero is approximately equal to an average of the integrals of adjacent polynomials or associated functions (degree $n-1$ and $n+1$) if the integration is restricted to the interval $(-\tilde{\Psi},\tilde{\Psi})$:

$$0 < \Psi_l \le \tilde{\Psi} < \pi - \Delta\Psi \qquad \Delta\Psi > 0$$

where Ψ_l prevents the integration interval from being too small, $\Delta\Psi$ prevents integration in the neighborhood of π, and it is (reasonably) assumed that the contributions of

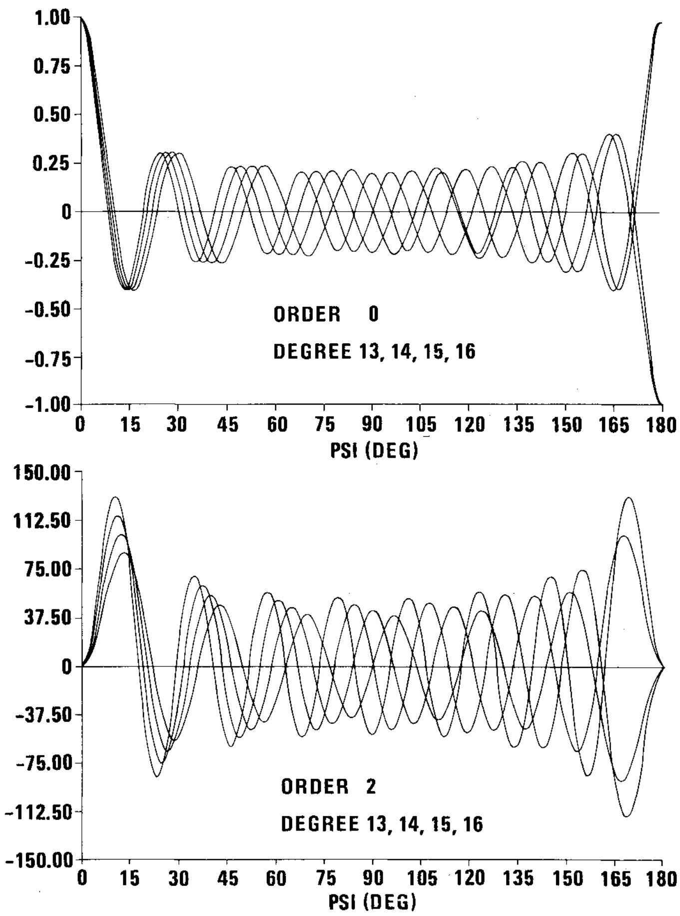

Fig. 3 Legendre functions.

diametrically located gravity modeling errors are negligible. The graphs of Fig. 3 lend intuitive support.

A demonstration for our second property is generated by insertion of the appropriate Legendre functions into $\rho(t)$, and use of a well-known Legendre recursion[17] into the left-hand side of Eq. (29) to remove the cosine factor at the expense of introducing the averaging requirement referred to above.

A. An Important Result

Correct modeling of gravity error auto-correlations is emphasized by considering the mapping of gravity acceleration errors into errors of orbital mean motion for a near-circular orbit. Since the in-track gravity acceleration errors are integrated to zero, the only contribution to orbital mean motion error (due to gravity errors) comes from radial gravity modeling errors. Thus mean motion induced positional in-track errors are due only to radial acceleration errors even though the instantaneous in-track acceleration errors have significant amplitude!

B. An Example

Returning to the example associated with Fig. 1, consider an orbit with period 100 min, eccentricity 0.02, argument of perigee and argument of orbital latitude both zero. The following triangular array of root variances and cross-correlations reflects an evaluation of Eq. (6) and then a transformation to the Gaussian orthonormal trajectory frame with conversion to convenient form:

22.46					
0.07	66.98				
0.00	0.00	22.08			
−0.07	−0.93	0.00	0.07		
−1.00	−0.07	0.00	0.07	0.02	
0.00	0.00	−0.01	0.00	0.00	0.03

The first three diagonal elements are radial, in-track, and cross-track positional root variances in units of feet. The last three diagonal elements are radial, in-track, and crosstrack velocity root variances in units of feet per second. The off-diagonal elements are cross-correlations ($t_{j+1} - t_j = 100$ min).

X. Conclusions

It has been demonstrated that the random gravity modeling errors associated with space vehicle orbits of small eccentricity cannot be physically represented with white process noise, and hence that the white process noise Kalman filter hypothesis should be rejected for use in sequential orbit determination. A sequential orbit determination algorithm is given which models auto-correlated gravity errors for near-circular orbits.

Acknowledgments

The methods of attack and solution to the problem defined herein evolved from a series of informal discussions between the author and W.F. Sampson. J. Curiale and W. Feess are thanked for enlightening discussions relating to gravity error auto-correlation.

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